JOURNAL OF APPROXIMATION THEORY 36, 257–264 (1982)

Polynomial Approximation Using Equioscillation on the Extreme Points of Chebyshev Polynomials

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Communicated by Richard S. Varga

Received September 3, 1981

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One method of obtaining near minimax polynomial approximation to $f \in C^{(n+1)}[-1, 1]$ is to choose $p \in \mathscr{P}_n$ such that $f-p$ equioscillates on the point set consisting of the extrema of T_{n+1}. It is shown that $\|f-p\|$ may be expressed in terms of $f^{(n+1)}$ in the same manner as $E_n(f)$, the error of minimax approximation. The Lebesgue constants are also investigated.

1. INTRODUCTION

Suppose $f \in C^{(n+1)}[-1, 1]$. Then it is known that for minimax polynomial approximation on $[-1, 1]$

$$E_n(f) := \min_{p \in \mathscr{P}_n} \|f-p\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\xi)|, \tag{1.1}$$

where $\xi \in (-1, 1)$ and $\|\cdot\|$ denotes the Chebyshev norm on $[-1, 1]$.

It is also well known that near minimax approximation is given by the interpolating polynomial $q \in \mathscr{P}_n$ constructed on the zeros of T_{n+1}, the Chebyshev polynomial of degree $n+1$. We then have

$$Z_n(f) := \|f-q\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\zeta)|, \tag{1.2}$$

0021-9045/82/110257–08$02.00/0

where $\zeta \in (-1, 1)$. Details of both of these results may be found, for example, in Phillips and Taylor [3]. It is also possible to find the Lebesgue constant $\Lambda_n(T)$ for such interpolation and deduce that

$$Z_n \leqslant (1 + \Lambda_n(T))\, E_n \leqslant \left(2 + \frac{2}{\pi} \ln (n+1)\right) E_n.$$

See Rivlin [4].

Another method of obtaining near minimax approximation is to choose the unique $p \in \mathscr{P}_n$ such that the error $f - p$ equioscillates on the point set consisting of the $n+2$ extrema of T_{n+1}. Such an approximation is usually suggested as a means of starting the Remez exchange algorithm. We will investigate the error $f - p$ and show that a result similar to (1.1) and (1.2) holds.

2. Equioscillation on Extrema of T_{n+1}

Let $\eta_j = \cos(j\pi/(n+1))$, $j = 0,\dots, n+1$, denote the extrema of T_{n+1} on $[-1, 1]$. We note that

$$\prod_{j=0}^{n+1} (x - \eta_j) = \frac{(x^2 - 1)\, U_n(x)}{2^n},$$

where $U_n \in \mathscr{P}_n$ is the Chebyshev polynomial of the second kind.

Suppose $q_{n+1} \in \mathscr{P}_{n+1}$ is the interpolatory polynomial for f constructed on $H = \{\eta_0,\dots, \eta_{n+1}\}$, when

$$f(x) - q_{n+1}(x) = \frac{(x^2 - 1)\, U_n(x)}{2^n} f[x, \eta_0,\dots, \eta_{n+1}].$$

The coefficient of x^{n+1} in $q_{n+1}(x)$ is $f[\eta_0,\dots, \eta_{n+1}]$ and we "economise" q_{n+1} to obtain

$$p(x) = q_{n+1}(x) - \frac{1}{2^n} f[\eta_0,\dots, \eta_{n+1}]\, T_{n+1}(x),$$

where $p \in \mathscr{P}_n$. It is clear that $f - p$ equioscillates on H as $f(\eta_j) - q_{n+1}(\eta_j) = 0$, $j = 0,\dots, n+1$.

We also have

$$f(x) - p(x) = \frac{(x^2 - 1)\, U_n(x)}{2^n} f[x, \eta_0,\dots, \eta_{n+1}] + \frac{T_{n+1}(x)}{2^n} f[\eta_0,\dots, \eta_{n+1}] \tag{2.1}$$

and observe that

$$\|f-p\| \geqslant |f(\eta_j)-p(\eta_j)| = \frac{1}{2^n}|f[\eta_0,..., \eta_{n+1}]|$$

$$= \frac{1}{2^n(n+1)!}|f^{(n+1)}(\lambda)| \geqslant \frac{m_{n+1}}{2^n(n+1)!}, \tag{2.2}$$

where $\lambda \in (-1, 1)$ and

$$m_{n+1} = \min_{x\in[-1,1]} |f^{(n+1)}(x)|.$$

THEOREM. *If* $f \in C^{(n+1)}[-1, 1]$ *and* $p \in \mathscr{P}_n$ *is such that* $f-p$ *equioscillates on* H *then*

$$\|f-p\| = \frac{1}{2^n(n+1)!}|f^{(n+1)}(\mu)|,$$

where $\mu \in (-1, 1)$.

Proof. We need to prove that

$$\|f-p\| \leqslant \frac{M_{n+1}}{2^n(n+1)!}, \tag{2.3}$$

where $M_{n+1} = \|f^{(n+1)}\|$. The result then follows from (2.2) and the continuity of $f^{(n+1)}$.

It is not possible to obtain (2.3) by simply bounding the two terms in (2.1) as their signs may differ. We let $x = \cos\theta$, $\theta \in [0, \pi]$, $e(x) = 2^n(f(x)-p(x))$ and write (2.1) as

$$e(x) = -\sin\theta \cdot \sin(n+1)\theta \cdot f[x, \eta_0,..., \eta_{n+1}]$$
$$+ \cos(n+1)\theta \cdot f[\eta_0,..., \eta_{n+1}]. \tag{2.4}$$

We consider four separate cases:

(i) $\sin(n+1)\theta$ *and* $\cos(n+\frac{1}{2})\theta$ *of opposite signs.*

We use

$$f[x, \eta_0,..., \eta_{n+1}] = \frac{f[x, \eta_0,..., \eta_n] - f[\eta_0,..., \eta_{n+1}]}{\cos\theta + 1} \tag{2.5}$$

to write (2.4) as

$$\begin{aligned} e(x) &= -\frac{\sin\theta \cdot \sin(n+1)\theta}{\cos\theta + 1} f[x, \eta_0, ..., \eta_n] \\ &\quad + \left(\cos(n+1)\theta + \frac{\sin\theta \cdot \sin(n+1)\theta}{\cos\theta + 1}\right) f[\eta_0, ..., \eta_{n+1}] \\ &= -\tan\tfrac{1}{2}\theta \cdot \sin(n+1)\theta \cdot f[x, \eta_0, ..., \eta_n] \\ &\quad + \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta} f[\eta_0, ..., \eta_{n+1}]. \end{aligned}$$

The factors $-\tan\frac{1}{2}\theta \cdot \sin(n+1)\theta$ and $\cos(n+\frac{1}{2})\theta/\cos\frac{1}{2}\theta$ are of the same sign and hence

$$|e(x)| \leqslant |\cos(n+1)\theta| \frac{M_{n+1}}{(n+1)!} \leqslant \frac{M_{n+1}}{(n+1)!}. \tag{2.6}$$

(ii) $\sin(n+1)\theta$ *and* $\sin(n+\frac{1}{2})\theta$ *of opposite signs.*

We use

$$f[x, \eta_0, ..., \eta_{n+1}] = \frac{f[x, \eta_1, ..., \eta_{n+1}] - f[\eta_0, ..., \eta_{n+1}]}{\cos\theta - 1} \tag{2.7}$$

to deduce that

$$\begin{aligned} e(x) &= \cot\tfrac{1}{2}\theta \cdot \sin(n+1)\theta \cdot f[x, \eta_1, ..., \eta_{n+1}] \\ &\quad - \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} f[\eta_0, ..., \eta_{n+1}] \end{aligned}$$

when again (2.6) follows.

(iii) $\cos(n+1)\theta$ *and* $\cos(n+\frac{1}{2})\theta$ *of opposite signs.*

We first note that

$$\begin{aligned} |f[x, \eta_0, ..., \eta_{n+1}]| &= \frac{1}{2} |f[x, \eta_1, ..., \eta_{n+1}] - f[x, \eta_0, ..., \eta_n]| \\ &\leqslant \frac{M_{n+1}}{(n+1)!} \end{aligned} \tag{2.8}$$

as $\eta_0 = 1$ and $\eta_{n+1} = -1$.

We then use (2.5) in (2.4) to deduce that

$$\begin{aligned} e(x) &= -(\cos n\theta + \cos(n+1)\,\theta)\cdot f[x, \eta_0,\ldots, \eta_{n+1}] \\ &\quad + \cos(n+1)\,\theta\cdot f[x, \eta_0,\ldots,\eta_n] \\ &= -2\cos(n+\tfrac{1}{2})\,\theta\cdot\cos\tfrac{1}{2}\theta\cdot f[x,\eta_0,\ldots,\eta_{n+1}] \\ &\quad + \cos(n+1)\,\theta\cdot f[x,\eta_0,\ldots,\eta_n]. \end{aligned}$$

The signs of the factors $-2\cos(n+\frac{1}{2})\,\theta\cdot\cos\frac{1}{2}\theta$ and $\cos(n+1)\,\theta$ are the same and hence, using also (2.8),

$$|e(x)| \leqslant |\cos n\theta|\cdot\frac{M_{n+1}}{(n+1)!} \leqslant \frac{M_{n+1}}{(n+1)!}. \tag{2.9}$$

(iv) $\cos(n+1)\,\theta$ *and* $\sin(n+\frac{1}{2})\,\theta$ *of same sign.*

We use (2.7) in (2.4) to deduce that

$$\begin{aligned} e(x) &= (\cos(n+1)\,\theta - \cos n\theta)\cdot f[x,\eta_0,\ldots,\eta_{n+1}] \\ &\quad + \cos(n+1)\,\theta\cdot f[x,\eta_1,\ldots,\eta_{n+1}] \\ &= -2\sin(n+\tfrac{1}{2})\cdot\sin\tfrac{1}{2}\theta\cdot f[x,\eta_0,\ldots,\eta_{n+1}] \\ &\quad + \cos(n+1)\,\theta\cdot f[x,\eta_1,\ldots,\eta_{n+1}]. \end{aligned}$$

This time the relevant factors have opposite signs and (2.9) follows.

Result (2.6) follows immediately from (2.4) for $\theta=0$ and $\theta=\pi$. For $\theta\in(0,\pi)$, it can be seen from Table I that (i)–(iv) cover all possibilities as $\frac{1}{2}\theta\in(0,\frac{1}{2}\pi)$. Thus the proof of the theorem is complete.

TABLE I

Quadrant which contains		
$(n+\frac{1}{2})\,\theta$	$(n+1)\,\theta$	Relevant case
1	1	(iv)
1	2	(iii)
2	2	(i)
2	3	(ii)
3	3	(iv)
3	4	(iii)
4	4	(i)
4	1	(ii)

3. Lebesgue Constant

Suppose $p \in \mathscr{P}_n$ is as defined in Section 2 and thus $f-p$ equioscillates on H. We may write p in terms of Chebyshev polynomials as follows.

$$p(x) = \sum_{k=0}^{n}{}' \, c_k T_k(x), \tag{3.1}$$

where

$$c_k = \frac{2}{n+1} \sum_{j=0}^{n+1}{}'' f(\eta_j) \cdot \cos \frac{kj\pi}{(n+1)}. \tag{3.2}$$

See Fraser [1]. ($\sum'$ denotes summation with the first term halved and $\sum''$ summation with the first and last terms halved.) Thus

$$p(\cos\theta) = \frac{2}{n+1} \sum_{k=0}^{n}{}' \sum_{j=0}^{n+1}{}'' f(\eta_j) \cdot \cos k\theta_j \cos k\theta$$
$$= \frac{1}{n+1} \sum_{j=0}^{n+1}{}'' f(\eta_j) \sum_{k=0}^{n}{}' \, [\cos k(\theta+\theta_j) + \cos k(\theta-\theta_j)],$$

where $\theta_j = j\pi/(n+1)$, $j = 0,\ldots, n+1$. Hence

$$\|p\| \leqslant \Lambda_n(H) \cdot \|f\|,$$

where $\Lambda_n(H)$, the Lebesgue constant of the mapping of f to p, is given by

$$\Lambda_n(H) = \max_{\theta\in[0,\pi]} h_n(\theta) = \max_{\theta\in[0,\pi/2]} h_n(\theta)$$

with

$$h_n(\theta) = \frac{1}{n+1} \sum_{j=0}^{n+1}{}'' \left| \sum_{k=0}^{n}{}' \, [\cos k(\theta+\theta_j) + \cos k(\theta-\theta_j)] \right|$$
$$= \frac{1}{2(n+1)} \sum_{j=0}^{n+1}{}'' \left| \frac{\sin[(n+\frac{1}{2})(\theta+\theta_j)]}{\sin\frac{1}{2}(\theta+\theta_j)} + \frac{\sin[(n+\frac{1}{2})(\theta-\theta_j)]}{\sin\frac{1}{2}(\theta-\theta_j)} \right|$$
$$= \frac{1}{2(n+1)} \sum_{j=0}^{n+1}{}'' \left| \sin(n+1)\theta \cdot \left(\cot\frac{1}{2}(\theta+\theta_j) + \cot\frac{1}{2}(\theta-\theta_j)\right) - 2\cos(n+1)\theta \right|$$
$$\leqslant \frac{|\sin(n+1)\theta|}{2(n+1)} \sum_{j=1}^{n+1}{}'' \left| \cot\frac{1}{2}(\theta+\theta_j) + \cot\frac{1}{2}(\theta-\theta_j) \right| + 1. \tag{3.3}$$

It now follows, making use of the bounds derived in McCabe and Phillips [2], that

$$\Lambda_n(H) \leqslant \Lambda_n(T) + 1, \tag{3.4}$$

where $\Lambda_n(T)$ is the Lebesgue constant for interpolation on the zeros of T_{n+1}. Values of $\Lambda_n(H)$ for various n were calculated and are given in Table II. $\theta_{\max}$ is the point at which h_n is a maximum. Values of $\Lambda_n(T)$ are also shown and it is conjectured that in place of (3.4) we have the stronger results,

$$\begin{aligned} \Lambda_n(H) &< \Lambda_n(T) && \text{for } n \text{ odd and } n > 1, \\ \Lambda_n(H) &= \Lambda_n(T) && \text{for } n \text{ even}, \\ \theta_{\max} &= \pi/2 && \text{for } n \text{ even}. \end{aligned}$$

TABLE II

n	$\theta_{\max}$	$\Lambda_n(H)$	$\Lambda_n(T)$
1	0	1.5	1.414
2	0 or $\frac{1}{2}\pi$	1.667	1.667
3	1.104	1.830	1.848
4	$\frac{1}{2}\pi$	1.989	1.989
5	1.284	2.094	2.104
6	$\frac{1}{2}\pi$	2.202	2.202
7	1.362	2.280	2.287
8	$\frac{1}{2}\pi$	2.362	2.362
9	1.407	2.424	2.429
10	$\frac{1}{2}\pi$	2.489	2.489
50	$\frac{1}{2}\pi$	3.466	3.466
100	$\frac{1}{2}\pi$	3.901	3.901

However, the authors have not been able to find proofs of these conjectures.

It is easily seen from (3.3) and McCabe and Phillips [2, Eqs. (16) and (20)] that

$$h_n(\pi/2) = \Lambda_n(T)$$

when n is even.

References

1. W. Fraser, A survey of methods of computing minimax and near-minimax polynomial approximations for functions of a single independent variable, *J. Assoc. Comput. Mach.* **12** (1965), 295–314.
2. J. H. McCabe and G. M. Phillips, On a certain class of Lebesgue constants, *BIT* **13** (1973), 434–442.
3. G. M. Phillips and P. J. Taylor, "Theory and Applications of Numerical Analysis," Academic Press, London, 1973.
4. T. J. Rivlin, "The Chebyshev Polynomials," Wiley, New York, 1974.